CLASSIFICATION OF PERIODIC SOLUTIONS IN A SINGLE DEGREE-OF-FREEDOM SYSTEM WITH BACKLASH

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ABSTRACT

In this paper a single degree-of-freedom system with backlash is studied for its periodic response. This system is modeled as a piecewise linear system with discontinuity in the net restoring force, caused by additional damping in the contact-zone. The periodic orbits are classified by their number of subspace boundary crossings and Floquet multipliers. For this classification, the known analytical solutions in the different subspaces are used in the multiple shooting algorithm and a continuation method. Some observations are also presented about the qualitative features (such as symmetry, rigid body solutions) exhibited by this class of systems.

1 INTRODUCTION

Clearance, dead zone or backlash is a common feature of many mechanical systems and can undermine the performance of the system. Backlash can be due to intended clearance necessary for assembly and operation. It can further be a result of operational wear and tear. The specific instances of appearance of backlash and its influence on the dynamics and control of systems includes power transmissions, robotics, measurement systems, manufacturing processes and structures. Backlash can lead to rattle and chaotic motion in gears in power trains which can lead to damage and noisy operation. Systems with backlash characteristics form a subclass of discontinuous mechanical systems. Backlash can be modeled as a discontinuity of the net restoring force (neglecting impact) with piecewise linear characteristics. Several researchers have investigated the effect of backlash on the dynamics which includes evaluation of bifurcations and chaos in gear systems subject to harmonic excitation [1], analysis of subharmonic resonances of an offshore structure as a bilinear oscillator model via simulation [2], evaluation of the rattling in torsional gear train models using harmonic balance methods [3], response analysis for such systems with parametric excitation [4], experimental and computational investigation of the global stability of the periodic response of single degree-of-freedom models with elastic stops [5], periodically forced piecewise linear oscillator [6], strongly nonlinear behavior of the oscillator in clearance [7], dynamics of the bi-linear oscillator [8] and oscillator with motion-limiting constraints [9].

In this paper a single degree-of-freedom system with backlash (neglecting impact) is analyzed for the effect of excitation parameters on the dynamics of the periodic response using the multiple shooting method [10]. Specifically, this paper presents new insights on the qualitative dynamics for a system which has infinite stiffness ratio between the stopper stiffness and the stiffness in the backlash region. This results in a classification of periodic orbits by their number of subspace boundary crossings and Floquet multipliers. A piecewise-linear stiffness and damping model is used which leads to a discontinuous jump in the net restoring force.

This paper is organized as follows: first, a model will be presented, which is followed by the analytical approach for computing the response (flow) and its integration into the multiple shooting method. Using this some results about the classification of the periodic orbits by their number of subspace boundary crossings and Floquet multipliers.
crossings will be presented. Finally, conclusions and recommendations will be given.

2 MODELING OF SYSTEM DYNAMICS

The single degree-of-freedom system with backlash is presented schematically in Fig. 1 and consists of a mass which can move freely between two stoppers. The dynamics of the stoppers is assumed to be fast enough to ensure that they return to their original position between successive contacts and are therefore at rest when a contact occurs. This assumption is only valid when the damping force is small in comparison to the spring force. This is verified for the system under consideration, so the dynamics of the stoppers do not have to be modeled and the equation of motion is given as:

\[ m\ddot{x} + C(x) + K(x) = F \]  

(1)

Here, \( m \) is the mass of the system, \( F \) denotes the external forcing and \( x = [x, \dot{x}]^T \) is the state vector. The restoring force \( K(x) \) and the damping force \( C(x) \) are given by:

\[
K(x) = \begin{cases} 
0, & x \in V \\
k_1(x+b), & x \in V_1 \\
k_2(x-b), & x \in V_2 
\end{cases}
\]  

(2)

\[
C(x) = \begin{cases} 
0, & x \in V \\
c_1x, & x \in V_1 \\
c_2x, & x \in V_2 
\end{cases}
\]  

(3)

The state space is divided into three subspaces \( V, V_1 \) and \( V_2 \) as is depicted in Fig. 2, based on contact or no-contact with the stoppers. As can be seen in Fig. 2, each boundary consists of two parts. When the mass moves towards a stopper (\( \dot{x} > 0 \)) it will hit it when \( |x| = b \), which explains the vertical parts of the boundary. However, the mass does not lose contact to the stopper when \( |x| = b \) again, but when the contact force becomes zero. Therefore the slope of the non-vertical parts are prescribed by the ratio of the spring and damper constant of the stopper.

This can mathematically be described as:

\[
V_1 = \{ x \in \mathbb{R}^2 | x < -b \land k_1(x+b) + c_1\dot{x} \leq 0 \} 
\]  

(4)

\[
V_2 = \{ x \in \mathbb{R}^2 | x > b \land k_2(x-b) + c_2\dot{x} \geq 0 \} 
\]  

(5)

These equations give the conditions for contact with a stopper. If the mass is in contact with the left stopper the state is in subspace \( V_1 \) whereas \( V_2 \) denotes contact with the right stopper.

When the mass is not in contact with a stopper the state is in subspace \( V \):

\[
V = \{ x \in \mathbb{R}^2 | x \notin (V_1 \cup V_2) \} 
\]  

(6)

In the backlash region no restoring force acts on the mass, only some damping force is present. The steady state forced response or periodic orbits of these systems is of interest as it dictates the long term dynamics and possible loss of stability for rotating machines such as geared systems. Often such systems operate at constant frequency where the forcing is given by:

\[
F = A \sin(\omega t) 
\]  

(7)

Using this forcing, the total equation can be written in first-order form as:

\[
x = f(t,x) = \left[ -\frac{1}{m} (K(x) + C(x)) + \frac{1}{m} A \sin(\omega t) \right] \dot{x} 
\]  

(8)

The nominal parameters are chosen to be \( m = 1 \) kg, \( c = 0.05 \) Ns/m, \( b = 1 \) m, \( k_1 = k_2 = 4 \) N/m, \( c_1 = c_2 = 0.5 \) Ns/m and \( A = 1 \) N.
3 ANALYSIS METHODOLOGY

3.1 Periodic orbits and stability - a review

In this work, periodic orbits and their stability are analyzed for different forcing parameters (frequency and amplitude). The flow of the nonautonomous system after a time lapse \( t_0 \) starting at \( t_0 \) is denoted by \( \Phi_0 (t_0, x_0) \equiv \Phi (t_0 + t, t_0, x_0) \). A periodic solution, \( \Phi^p (t, t_0, x_0) \), where \( T > 0 \) is the minimal period time, is defined as follows:

\[
\Phi^p (t + T, t_0, x_0) = \Phi^p (t, t_0, x_0) \quad \forall t
\]  

(9)

In this paper, only period-one periodic orbits are considered, so \( T = 2\pi / \omega \). The stability of a periodic orbit can be determined along a periodic orbit. For smooth systems, the changes subspace. This effect of this jump on the perturbation \( \Delta \Phi \) is discontinuous as well and exhibits jumps whenever the state changes subspace. This effect of this jump on the perturbation can be described by a saltation matrix \( \mathbf{S} \):

\[
\Delta \mathbf{x}(t_0 + \Delta t) = \Phi_M(t_0, x(t_0)) \Delta \mathbf{x}(t_0)
\]  

(10)

The monodromy matrix \( \Phi_M \) [11] is the fundamental solution matrix of the system matrix \( \Phi_M = \exp (\mathbf{A} t) \). For smooth systems, the matrix is stable, and the period orbit can be determined by calculating the eigenvalues of the monodromy matrix, which are called Floquet multipliers.

Since system (1) is discontinuous, the monodromy matrix is discontinuous as well and exhibits jumps whenever the state changes subspace. This effect of this jump on the perturbation can be described by a saltation matrix \( \mathbf{S} \) [11]:

\[
\mathbf{S} = \mathbf{I} + \left( \frac{\mathbf{f}_{p_2} - \mathbf{f}_{p_1}}{\mathbf{n}_{p_2}} \right) \mathbf{n}_{p_2}^T
\]  

(11)

In this equation, \( \mathbf{f}_{p_2} \) is the direction of the vectorfield along the solution just before the subspace boundary crossing, \( \mathbf{f}_{p_1} \) is the similar direction just after the crossing. The normal of the subspace boundary, at which the flow crosses, is denoted by \( \mathbf{n} \).

3.2 Simulation

The solution of the forced system can be obtained by integrating Eqn. (8). However, to get an accurate solution near a subspace boundary the solution tolerance must be low, causing a long simulation time. This simulation time can be reduced by utilizing an analytical solution as discussed next. Since the system model is piecewise-linear, an analytical solution can be calculated in each subspace. The standard description of a linear system is considered:

\[
\dot{x} = \mathbf{A}_t x + \mathbf{B} u(t)
\]  

(12)

Here, \( \mathbf{A}_t \) is the system matrix in subspace \( v \), \( \mathbf{B} \) is the input matrix and \( u(t) \) the input. For ease of notation, the subscript \( v \) indicating the subspace will be omitted from now onwards. Using the spectral decomposition \( \mathbf{A} = \mathbf{M} \Lambda \mathbf{M}^{-1} \) and generalized coordinates \( \mathbf{p} = \mathbf{M}^{-1} \mathbf{x} \), the solution for \( \mathbf{p} \) is given by:

\[
\mathbf{p} = e^{\mathbf{A} t} \mathbf{p}_0 + \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{M}^{-1} \mathbf{B} u(\tau) d\tau
\]  

(13)

Simplifying notation (with \( \hat{\mathbf{B}} = \mathbf{M}^{-1} \mathbf{B} \)) yields:

\[
\mathbf{p} = e^{\mathbf{A} t} (\mathbf{p}_0 + \mathbf{d}) \quad \mathbf{d} = \int_0^t e^{-\mathbf{A} \tau} \hat{\mathbf{B}} u(\tau) d\tau
\]  

(14)

It is assumed that the eigenvalues are distinct, so \( e^{\mathbf{A}(t-\tau)} \) is diagonal, which yields \( n \) decoupled equations. If the complex notation for the input \( u(\tau) = A \sin(\omega \tau + \phi) \) is used, the \( k \)-th entry in \( \mathbf{d} (\mathbf{d}_k) \) is given as:

\[
\mathbf{d}_k = \frac{A \hat{\mathbf{B}}_k}{2i} \int_0^t \left( e^{-(\lambda_k - i\omega) \tau} e^{i\phi} - e^{-(\lambda_k + i\omega) \tau} e^{-i\phi} \right) d\tau
\]  

(15)

Here, \( \hat{\mathbf{B}}_k \) is the \( k \)-th entry in the vector \( \hat{\mathbf{B}} \), \( \lambda_k \) is the \( k \)-th eigenvalue. The solution of (15) is:

\[
\mathbf{d}_k = \frac{A \hat{\mathbf{B}}_k}{2i} \left( -\left( e^{-(\lambda_k - i\omega) \tau} \right. \right. \left. \left. \left. -1 \right) e^{i\phi} \right. \right. \left. \left. + \left( e^{-(\lambda_k + i\omega) \tau} \right. \right. \left. \left. -1 \right) e^{-i\phi} \right) \right.
\]  

(16)

The solution for \( \mathbf{x} \) can then be obtained by \( \mathbf{x} = \mathbf{M} \mathbf{p} \). The algorithm calculating the solution of the piecewise-linear system, of which the flowchart is given in Fig. 3, starts by creating a time vector \( t \). Next, the subspace in which the initial condition \( x_0 \) is located is determined using Eqns. (4) to (6). Using this subspace \( v_0 \), the corresponding system matrices are selected and the solution for the entire time vector is calculated, without changing the matrices. For this data, the subspaces \( v_k \) at each \( t_k \) are calculated. When all \( v_k \) are equal, the solution does not leave the initial subspace and the total solution is found.

In general, solutions will exist on all three subspaces and hence not all \( v_k \) will be equal. In those cases, if \( v_{k+1} \) is the first point that differs from \( v_k \), the change of subspace is known to occur between corresponding locations in the time vector, \( t_k \) and \( t_{k+1} \). The switching time \( t_k \) can then be found at any arbitrary accuracy by calculating additional states and corresponding subspaces for \( t_k < t < t_{k+1} \).
When the switching time \( t_s \) is found, the new solution for
\( v_k, k = I + 1, I + 2, \ldots \) can be calculated using the new system
matrices and \( x_k \) as initial condition. The phase angle of the forcing
is repeated until the correct states are calculated for all times in
the time vector \( t \).

It should to be noticed that the time step, \( \Delta t = t_{k+1} - t_k \), has to be
chosen small enough to ensure that subspaces are not crossed
without calculating any data point in it. When this happens, the
change in subspace will not be noticed, resulting in an incorrect
solution. Since an analytical solution can be calculated for a large
number of points instantaneously, choosing a sufficiently small
\( \Delta t \) does not lead to an excessive increase in calculation time.

An advantage of this method is that it calculates the sub-
space boundary crossing times for a periodic orbit. This information
can be used to analytically calculate the overall fundamental
solution matrix \( \Phi_M \) by multiplication of individual fundamental
solution matrices in the different subspaces and the appropriate
salutation matrices \( S \) to describe the subspace boundary crossing.
This reduces the computation time for the multiple shooting al-
gorithm, which is presented next.

### 3.3 Multiple shooting

To estimate periodic orbits, the multiple shooting method is
used. This method is preferred to the single shooting method
because it uses a number of initial points along the periodic so-
lution instead of only one as shown in Fig. 4. The multiple
shooting method is therefore more robust. The \( N \) shooting points
are equally spaced in time with constant time step \( h = T/N \), so
\( t_k = t_0 + kh \), and are stored in the vector \( X = [x_1, x_2, \ldots, x_N]^T \).

The segment connecting point \( x_{k-1} \) to the next point \( x_k \) is given by:

\[
x_k = \Phi_h(t_k, x_{k-1})
\]  

(17)

Here, \( \Phi_h(t_{k-1}, x_{k-1}) \) denotes the solution of \( \dot{x}(t) = f(t, x(t)) \)
at time \( t_k \) starting at initial condition \( x_{k-1} \) (at \( t_{k-1} \)). This solution
is evaluated using the analytical procedure described above.
It can be seen in Fig. 4 that a periodic solution is found if all
segments connect, so when Eqn. (17) holds for \( k = 2, \ldots, N \) and
\( x_1 = \Phi_h(t_N, x_N) \). Therefore, a zero of the following function has
to be calculated:

\[
H(X) = \begin{bmatrix}
-x_1 + \Phi_h(t_N, x_N) \\
\vdots \\
-x_k + \Phi_h(t_{k-1}, x_{k-1}) \\
\vdots \\
-x_N + \Phi_h(t_{N-1}, x_{N-1}) 
\end{bmatrix}
\]  

(18)

The Newton-Raphson algorithm is used iteratively to obtain
an updated estimate of the periodic solution:

\[
\frac{\partial H}{\partial X} \Delta X = -H(X)
\]  

(19)

Here, the Jacobian is given as:

\[
\frac{\partial H}{\partial X} = \begin{bmatrix}
-I & 0 & \ldots & 0 & \Phi_h(t_N, x_N) \\
\Phi_h(t_1, x_1) - I & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \Phi_h(t_{N-1}, x_{N-1}) & -I
\end{bmatrix}
\]  

(20)

Here, \( \Phi_h(t_k, x_k) \) denotes the fundamental solution matrix at
time \( t_k + h \) for a solution with initial condition \( x_k \) at \( t_k \). When the

Figure 4: THE MULTIPLE SHOOTING METHOD
set of equations (19) is solved, the next iterate can be calculated by \( X^{(i+1)} = X^{(i)} + \Delta X^{(i)} \).

When the multiple shooting method is applied to the system with backlash in Fig. 1 (and described by Eqn. (8)), a problem arises for periodic solutions that are entirely in the backlash gap (and do not hit the stoppers). Because of the absence of stiffness in this region, a small perturbation in the position \( \Delta x(t_0) \) of the mass will neither grow or decay. The entire periodic orbit will just be shifted in position. The velocity will not be affected. This knowledge gives some insight in the monodromy matrix for this situation. Therefore, the perturbation \( \Delta x(t_0) = [\Delta x(t_0)^T 0]^T \) is considered. The monodromy matrix maps this perturbation \( \Delta x(t_0) \) to \( \Delta x(t_a + T) \), which is equal to the initial perturbation.

\[
\begin{bmatrix}
\Delta x(t_a + T) \\
0
\end{bmatrix} = \Phi_T 
\begin{bmatrix}
\Delta x(t_a) \\
0
\end{bmatrix} \tag{21}
\]

By inspecting this equation, it can be seen that the first column of the monodromy matrix is \( e_1 = [1 \ 0]^T \). This holds for all \( t_a \). As an example, this column is substituted into Eqn. (20) for a multiple shooting algorithm using three points \((N = 3)\). Then, columns 1, 3 and 5 of \( \partial \mathbf{H} / \partial \mathbf{X} \) are respectively:

\[
\begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}^T \tag{22}
\]

It is clear that these columns are not linearly independent, so \( \partial \mathbf{H} / \partial \mathbf{X} \) will not have full rank and Eqn. (19) can not be solved. This is a result of the rigid body motion possible in the backlash region. Thus, a rigid body constraint can be added to make the above matrix, \( \partial \mathbf{H} / \partial \mathbf{X} \), full rank. For \( N = 3 \), this equation is:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix} \Delta x^{(i)} = -\left( x_1^{(i)} + x_3^{(i)} + x_5^{(i)} \right) / 3 \tag{23}
\]

Here, \( x_1 \) denotes the first entry in the vector \( \mathbf{X} \), which is the position coordinate of the first shooting point. Similarly, \( x_3 \) and \( x_5 \) denote the position coordinates of the other shooting points. This equation basically ensures that the periodic solution is (roughly) located in the center between the two stoppers and does not influence the periodic solution itself. By adding this rigid body constraint, the total number of equations is one larger than the number of variables and thus a least squares solution can be calculated.

Since this extra equation is not needed when the periodic solution comes in contact with the stoppers, it is only used when the condition number of \( \partial \mathbf{H} / \partial \mathbf{X} \) is very high.

4 RESULTS

The dynamics of the system (Fig. 1) as described by Eqn. (8) is characterized by the response diagram in Fig. 5. This figure shows the amplitude of the periodic solution for a range of forcing frequencies \( \omega \) for nominal excitation amplitude, \( A = 1 \) N. Stable branches are indicated by solid lines, while unstable branches are shown by dashed lines. The amplitude of the periodic solution (amp) is defined as half the difference between the maximum and minimum position during one period. The branches are calculated using the multiple shooting algorithm as described in this paper in combination with continuation. It is clear that multiple solutions exist near the primary peak. The bending to the right of the primary peak in Fig. 5 is characteristic for a hardening oscillator.
The two stable branches containing A and C are connected by an

The distance between the stoppers is \( 2b \), so this linear solution will exist for amplitudes up to one, which is around \( \omega = 1 \) \( \text{rad/s} \). However, in the frequency range of \((1, 1.42)\) \( \text{rad/s} \), the backlash system has multiple solutions for the same excitation frequency, of which orbits A, B and C are an example. For \( \omega = 1.4 \) \( \text{rad/s} \), the solution at C is shown in Fig. 8. In this case the stoppers are engaged and the solution visits all subspaces. The two stable branches containing A and C are connected by an unstable branch. Orbit B in Fig. 8 is an example of an unstable periodic orbit on this branch. This unstable orbit also visits all subspaces.

When the top branch is tracked for decreasing excitation frequency, it loses stability at \( \omega = 0.7 \) \( \text{rad/s} \). However, a branch of asymmetrical periodic orbits originates at the same point. The asymmetrical periodic orbit at point D is depicted in Fig. 9. Since \( f(t, x) = -f(t, -x) \), at point D a version of the orbit mirrored in the origin can also be found.

For decreasing excitation frequency, the number of boundary crossings increases and hence can be used to classify the periodic orbits. Figure 10 shows examples of multiple boundary crossings. Each time the number of boundary crossings changes a corner collision bifurcation [12] takes place. In corner collision bifurcations the periodic solution just touches the subspace boundary at the discontinuity in the boundary prior to crossing the boundary for some change in parameter. In this system this is exhibited in the parameter space of excitation amplitude and frequency. It should be noted that the response diagram may be incomplete for this low-frequency region, but the focus of this paper is on the main branch.

Figure 6 shows the magnitude of the Floquet multipliers corresponding to the response diagram in Fig. 5. The branch where the response amplitude is smaller than one shows a Floquet multiplier equal to one, which is caused by the absence of a restoring

<table>
<thead>
<tr>
<th>Label</th>
<th>Floquet multipliers</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>1.0000 0.7990</td>
</tr>
<tr>
<td>B</td>
<td>1.4210 0.1482</td>
</tr>
<tr>
<td>C</td>
<td>0.6086 0.3085</td>
</tr>
<tr>
<td>D</td>
<td>-0.2249 -0.2261i</td>
</tr>
<tr>
<td>E</td>
<td>-0.0986 -0.0120i</td>
</tr>
<tr>
<td>F</td>
<td>0.0001 0.0</td>
</tr>
</tbody>
</table>

4.1 Periodic response

Next, some periodic orbits on the response curve in Fig. 5 are highlighted to discuss the characteristics of periodic orbits exhibited by this system. For frequencies \( \omega > 1 \) \( \text{rad/s} \), for some initial conditions, the mass can move in the region between the stoppers without hitting them. Examples of such an orbit are given in Fig. 7. Since the subspace boundaries are not crossed by these periodic solutions, the dynamics are purely linear. In this subspace, no restoring force is present. The periodic orbit is therefore not unique; it can be shifted in position. This however does not affect the nature of the periodic solution and the amplitude will not change by shifting the solution. Because of the absence of a restoring force, one Floquet multiplier is equal to one, as is shown in Tab. 1. The Floquet multipliers are not affected by a shift of the periodic solution in the backlash region.

The Floquet multipliers for the labeled periodic orbits in Fig. 5 are given in Table 1. The multipliers equal to one, which is caused by the absence of a restoring

### Table 1: FLOQUET MULTIPLIERS FOR THE LABELED PERIODIC ORBITS IN FIG. 5

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</tr>
<tr>
<td>F</td>
<td>0.0001 0.0</td>
</tr>
</tbody>
</table>

Figure 7: PERIODIC ORBITS AT POINT A IN FIG. 5 (\( \omega = 1.4 \) \( \text{rad/s} \))

Figure 8: PERIODIC ORBITS AT POINTS B (LEFT) AND C (RIGHT) IN FIG. 5 (\( \omega = 1.4 \) \( \text{rad/s} \))

Figure 9: PERIODIC ORBITS AT POINT D IN FIG. 5 (\( \omega = 0.65 \) \( \text{rad/s} \))
force. At the point where this branch crosses the boundary to become an unstable orbit a discontinuous fold bifurcation [13] occurs. Characteristic for this bifurcation is the jump of Floquet multipliers through the unit circle as can be observed in Fig. 6 near $\omega = 1$ rad/s. The point near $\omega = 0.7$ rad/s where the stable symmetrical branch splits into an unstable symmetrical and stable asymmetrical branch can be clearly recognized. It has to be noted that for a range of frequencies between 0.42 and 0.54 rad/s no stable period one solutions exist. This can also be concluded from Fig. 11, which shows a bifurcation diagram for a forcing amplitude $A = 1$ N. This figure suggests chaotic or quasi-periodic behavior in this frequency range.

### 4.2 Boundary Classification

The dynamics of the system can be characterized by counting the number of boundary crossings of a periodic orbit. A boundary crossing is counted every time the periodic solution changes subspace. This is done for different forcing frequencies and amplitudes, yielding Figs. 12 and 15. The periodic solutions are found by using the multiple shooting algorithm. For Fig. 12, the initial condition for the simulation is calculated using the linear system description in subspace $V$ to ensure that the solution stays in the center between the stoppers when possible. This will result in solutions on the lower branch of Fig. 5. On the contrary, the initial condition (for the simulation) for Fig. 15 is chosen to be in a region where there is contact with a stopper, therefore increasing the probability of finding the solutions on the top branch of Fig. 5.

Based on the linear equations in subspace $V$, the dashed line, (see Figs. 12 and 15), where the amplitude of the periodic response is equal to one can be calculated analytically. In this case the periodic solution just touches (but does not cross) the boundary. This is referred to as a corner collision boundary [12].

The basic trend in Fig. 12 is that the number of crossings increases for decreasing frequency. For low frequencies, the direction of the force stays the same for a longer time span. Here, the dynamics of the system in contact with the stoppers is faster than the change in external forcing direction. The graph also shows that the numerically calculated boundary, that indicates the conditions where the mass first hits the stoppers, matches the analytically calculated boundary very well. The small discrepancy is likely to be caused by the simulated solution not to be exactly in the center between the stoppers.

The non-periodic region, with excitation frequencies between 0.42 and 0.54 rad/s for $A = 1$ N, that was shown in Fig. 11 can also be observed in Fig. 12. No stable solutions exist in this region labeled $d$ in the figure.

For all periodic orbits that are classified by their number of boundary crossings the Floquet multipliers are calculated. The Floquet multipler with the maximum absolute value is depicted in Figs. 13 and 16. Black and white denote an absolute value of zero and one respectively. All (unstable) Floquet multipliers with an absolute value higher than one are set to one for clarity. This figure also clearly shows the boundaries. This can be explained by considering the monodromy matrices. Each time a switching boundary, in the phase plane, is crossed the monodromy matrix exhibits a discontinuity or jump. This jump (which is described by a saltation matrix) also affects the Floquet multipliers, so a change in number of boundary crossings will also cause a sudden change in Floquet multipliers.

This number of boundary crossing is not the only qualitative difference in the periodic solutions. Fig. 13, when compared to Fig. 12 shows an extra boundary between labels A and B. This suggests a change in the characteristics of the periodic orbit, although the number of crossings does not change. This idea is verified in Fig. 14 (a) to (c), which shows the periodic orbits at labels A, B and C. The periodic orbit changes via a symmetry breaking bifurcation from being symmetric at A to asymmetric at
Figure 12: CLASSIFICATION OF PERIODIC ORBITS IN \((A, \omega)\) SPACE: BOUNDARY CROSSINGS

Figure 13: CLASSIFICATION OF PERIODIC ORBITS IN \((A, \omega)\) SPACE: FLOQUET MULTIPLIER WITH LARGEST MODULUS
the number of crossings. The number of boundary crossings for is again asymmetric. This could also have been concluded from Figures 8 and 9, when the response diagram was discussed. The symmetric and asymmetric periodic orbits were already shown in Figs. 12 and 13 when the response exhibits more than four boundary crossings. The diagrams are different at the right side of the corner collision boundary. The region where the periodic solutions cross the boundaries four times is increased. A fractal-like boundary can be seen where this region ends. At the right side of that boundary two situations occur: first, there are periodic orbits in V that do not cross any boundary, as was observed earlier and second, periodic orbits with two boundary crossings are found. This means that only one stopper is touched; Fig. 16 indicates that these periodic orbits are stable. Depending on the initial condition of the multiple shooting algorithm, either this solution or the non-touching solution is found.

5 CONCLUSIONS AND FUTURE WORK

In this paper, preliminary results are presented on the classification of the periodic orbits associated with a single degree-of-freedom system with backlash. The periodic orbits are evaluated using a simulation based method that uses the analytical solution in the different subspaces. Since it also calculates boundary crossing times, the monodromy matrix can be analytically calculated as well. For this, fundamental solution and saltation matrices are used. This is then integrated into the multiple shooting method. It is shown that the Floquet multipliers undergo a sudden change when the number of crossings of a periodic orbit changes. Floquet multipliers thus give the same classification as B, without a change in number of boundary crossings. Both the symmetric and asymmetric periodic orbits were already shown in Figs. 8 and 9, when the response diagram was discussed. The asymmetrical periodic orbit at point C does show extra boundary crossings. This change can be noticed in both Fig. 12 and 13.

Figure 14 (d) to (f) show the periodic orbits at points D, E and F, which show an increase in the number of crossings as the forcing frequency decreases. The periodic orbit at point E is again asymmetric. This could also have been concluded from the number of crossings. The number of boundary crossings for this periodic orbit is ten, which means the stoppers are hit five times in a period. Since five is odd, the periodic orbit has to be asymmetrical. It has to be noticed that a number of boundary crossings which is a multiple of four does not mean that the periodic orbit is symmetrical, as can be observed by considering the periodic orbit at point B in Fig. 14 (b).

For low excitation frequency (ω < 0.2 rad/s), the amplitude of the forcing appears to have a larger influence on the number of crossings than the frequency. Figure 14 (g) and (h) shows this influence for ω = 0.15 rad/s. The periodic orbit at G shows higher harmonics that are entirely in subspace V₁ or V₂, so contact with a stopper is not lost. For a lower forcing amplitude, the force is too small to maintain this and contact with the stopper will be lost, causing an increase in the number of boundary crossings. Periodic orbit H in Fig. 14 (h) is an example.

Since Figs. 12 and 13 are created using initial conditions to force the periodic orbit to be in the center between the stoppers when possible, a part of the top branch which overlaps the linear branch of the response diagram (Fig. 5) is not found. Figures 12 and 13 are therefore recreated using a different initial condition (x₀ = [3 0]T) to enforce the possibility of finding periodic orbits on that part of the top branch of the response diagram. The result is depicted in Figs. 15 and 16, where the latter again shows the maximum absolute value of the Floquet multipliers.

The analytically calculated corner collision boundary does not depend on the initial conditions. Further, the solutions and their characteristics in Figs. 15 and 16 are almost identical to Figs. 12 and 13 when the response exhibits more than four boundary crossings. The diagrams are different at the right side of the corner collision boundary. The region where the periodic solutions cross the boundaries four times is increased. A fractal-like boundary can be seen where this region ends. At the right side of that boundary two situations occur: first, there are periodic orbits in V that do not cross any boundary, as was observed earlier and second, periodic orbits with two boundary crossings are found. This means that only one stopper is touched; Fig. 16 indicates that these periodic orbits are stable. Depending on the initial condition of the multiple shooting algorithm, either this solution or the non-touching solution is found.

Figure 14: PERIODIC ORBITS AT THE LABELS IN FIG. 12
the number of subspace boundary crossings. Next, Floquet multipliers give some additional information on symmetry. Specifically, the parameter space of excitation frequency and amplitude is classified via the boundary collision bifurcations and symmetry breaking bifurcations.

This classification of periodic orbits will be extended for a more realistic, multiple degree-of-freedom system model with backlash as part of the future work related to this research.

REFERENCES